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SOME NOTES ON RELATIVELY COMPACT SUBSETS OF FUZZY SETS

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Abstract. In this note, we deal with some characterizations of relative compactness on the L_p metric space of fuzzy sets. And then, we point out that a characterization of relative compact subsets of fuzzy numbers with sendograph metric can be improved.

1. Introduction

The metric in a space of fuzzy sets plays an important role both in the theory and in its applications. There are various useful metrics defined on the fuzzy number space $\mathbf{E}^n$ of normal, upper-semicontinuous, compact-supported and convex fuzzy subset of n-dimensional Euclidean space R^n. The readers may refer to [2] for supremum metric, sendograph metric and L_p-metric, and refer to [7] for Skorohod metric.

It is well-known that $\mathbf{E}^n$ is complete and separable if it is equipped with the metric except L_p-metric. Characterizations of compact subsets of $\mathbf{E}^n$ equipped supremum metric, sendograph metric and the Skorohod metric were given by Greco [5], Greco and Moschen [6], Greco [4], Wu and Zhao [11], Joo and Kim [7], respectively.

However, it is known that $\mathbf{E}^n$ is separable but not complete with respect to the L_p-metric. Because of this fact, compactness criteria in the space $\mathbf{E}^n$ equipped with the L_p-metric is given by a little complicated form. A characterization of compact subsets of the space $\mathbf{E}^n$ equipped with the L_p-metric was firstly given by Diamond and Kloeden [2]. Later on, Ming [10] indicated an error of [2] and suggested a modified characterization of compact sets. Wu and Zhao [11] gave an counter-example

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which shows that the modified characterization in [10] is incorrect, and established correct characterization.

For relative compactness with respect to sendograph metric, Fan [3] characterized relative compact subsets of $\mathbf{E}^1$, Greco [4] generalized the result of Fan [3] to arbitrary metric space. And then, Zhao and Wu [12] gave a new characterization of relative compact subsets of fuzzy sets in R^n.

In this note, we first discuss with relative compactness on L_p metric space of fuzzy sets, and then point out that a new characterization of relative compactness with respect to sendograph metric obtained by Zhao and Wu [12] can be improved.

2. Preliminaries

Let $\mathbf{K}(R^n)$ denote the family of all non-empty compact subsets of the n-dimensional Euclidean space R^n with the usual norm $|\cdot|$. Then the space $\mathbf{K}(R^n)$ is metrizable by the Hausdorff metric h defined by

$$h(A,B) \ = \ \max[\sup_{a\in A} \inf_{b\in B} |a-b|, \sup_{b\in B} \inf_{a\in A} |a-b|].$$

The norm of $A \in \mathbf{K}(R^n)$ is defined by

$$\|A\| = h(A,\{0\}) = \sup_{a\in A} |a|.$$

It is well-known that $\mathbf{K}(R^n)$ is complete and separable with respect to the Hausdorff metric h. Also, if we denote by $\mathbf{K}_c(R^n)$ the family of all $A \in \mathbf{K}(R^n)$ which is convex, then $\mathbf{K}_c(R^n)$ is a closed subspace of $(\mathbf{K}(R^n), h)$.

Let $\mathbf{F}(R^n)$ denote the family of all fuzzy sets $u : R^n \to [0,1]$ with the following properties;

(i) u is normal, i.e., there exists $x \in R^n$ such that $u(x) = 1$.
(ii) $L_\alpha u = \{x \in R^n : u(x) \geq \alpha\}$ is a compact subset of R^n for each $0 < \alpha \leq 1$.

$L_\alpha u$ is called the α-level set of u. We denote by $\mathbf{F}_c(R^n)$ the family of all $u \in \mathbf{F}(R^n)$ which is convex, i.e., $u(\lambda x + (1-\lambda)y) \geq \min(u(x), u(y))$ for all $x, y \in R^n$ and $0 \leq \lambda \leq 1$. Then $u \in \mathbf{F}_c(R^n)$ if and only if $L_\alpha u \in \mathbf{K}_c(R^n)$ for each $0 < \alpha \leq 1$.

Also, we denote by $\mathbf{F}_\infty(R^n)$ (resp. $\mathbf{F}_{c,\infty}(R^n)$) the family of all $u \in \mathbf{F}(R^n)$ (resp. $\mathbf{F}_c(R^n)$) with compact support, i.e.,

$$L_0 u = \overline{\{x \in R^n : u(x) > 0\}}$$

is compact, where $\overline{A}$ denotes the closure of A with respect to the usual norm in R^n. Briefly, $\mathbf{F}_{c,\infty}(R^n)$ is denoted by $\mathbf{E}^n$ and a member of $\mathbf{E}^n$ is called a fuzzy number.

The uniform metric d_∞ on $\mathbf{F}_\infty(R^n)$ is defined as usual;

$$d_\infty(u,v) = \sup_{0\leq\alpha\leq 1} h(L_\alpha u, L_\alpha v).$$

Also, the norm of u is defined as

$$\|u\| = d_\infty(u,\tilde{0}) = \|L_0 u\| = \sup_{x\in L_0 u} |x|,$$

where $\tilde{0}$ denotes the indicator function of $\{0\}$.

Another metrics on $\mathbf{F}_\infty(R^n)$ can be defined as follows;

$$\begin{aligned} d_p(u,v) &= (\int_0^1 h(L_\alpha u, L_\alpha v)^p \, d\alpha)^{1/p}, \ 1 \leq p < \infty \\ D(u,v) &= h^*(send(u), send(v)), \end{aligned}$$

where $send(u) = \{(x,\alpha) \in R^n \times [0,1] : x \in L_0 u$ and u(x) $\geq \alpha\}$ is the sendograph of u and and h^* is the Hausdorff metric metric on $R^n \times [0,1]$.

The metrics d_p and D are called the L_p-metric and sendograph metric, respectively. Relations for convergence of fuzzy sets with respect to these metrics can be found in Diamond and Kloeden [2].

3. Results

We first discuss with the relative compactness on $(\mathbf{F}_\infty(R^n), d_p)$. As a matter of fact, Wu and Zhao [10] characterized the relative compactness on $(\mathbf{E}^n, d_p)$. Their result is also valid in the case of $(\mathbf{F}_\infty(R^n), d_p)$. To review the result, let us denote $u^{(r)} = uI_{L_r u}$, i.e.,

$$u^{(r)}(x) = \begin{cases} u(x) & \text{if} \quad u(x) \geq r, \\ 0 & \text{if} \quad u(x) < r, \end{cases}$$

for $u \in \mathbf{F}(R^n)$ and $0 < r < 1$.

Then the result of Wu and Zhao [11] can be modified as following;

Theorem 3.1. *$\mathcal{U} \subset \mathbf{F}_\infty(R^n)$ is relatively compact in $(\mathbf{F}_\infty(R^n), d_p)$ if and only if*

(3.1) *$\mathcal{U}$ is uniformly p-th mean bounded, i.e.,* $\sup_{u\in U} \int_0^1 \|L_\alpha u\|^p \, d\alpha < \infty$.

(3.2) *$\mathcal{U}$ is p-th mean equi-left-continuous, i.e.,*

$$\lim_{\delta\to 0}\sup_{u\in U}\int_\delta^1 h(L_\alpha u, L_{\alpha-\delta}u)^p\, d\alpha \;=\; 0.$$

(3.3) *Let $\{r_i\}$ be a decreasing sequence in $(0,1]$ converging to 0 and $\{u_k\}$ be a sequence in $\mathcal{U}$ such that $\{u_k^{(r_i)}\}$ converges to $v_{r_i}\in \mathbf{F}_\infty(R^n)$ in d_p for each r_i. Then there exists a $v\in\mathbf{F}_\infty(R^n)$ such that $L_\alpha v = L_\alpha v_{r_i}$ for all $\alpha\in(r_i,1]$ and all r_i,*

In fact, (3.3) is necessary because $(\mathbf{F}_\infty(R^n), d_p)$ is not complete. First we show that (3.3) is equivalent to that the closure of $\mathcal{U}$ in $(\mathbf{F}_\infty(R^n), d_p)$ is complete. To this end, we concern about the completion of $(\mathbf{F}_\infty(R^n), d_p)$. It is trivial that such a completion exists by well-known facts in Analysis. Related to this problem, Kraschmer [9] dealt with completion of $\mathbf{E}^n$ with respect to the L_p-metric by introducing the notion of support function for noncompact fuzzy number, and Degang et al. [1] proposed the completion of $\mathbf{E}^1$ with respect to the L_1-metric by using representation theorem of noncompact fuzzy number in R. But these approaches are available only if we assume the convexity condition. Nevertheless, similar results can be obtained in a direct manner. To describe the result, let $\mathbf{F}_p(R^n)$ (resp. $\mathbf{F}_{c,p}(R^n)$) be the family of all fuzzy sets $u\in\mathbf{F}(R^n)$ (resp. $\mathbf{F}_c(R^n)$) such that

$$\int_0^1 \|L_\alpha u\|^p\, d\alpha \;<\; \infty,$$

for $1\le p<\infty$. Then it is obvious that the d_p on $\mathbf{F}_p(R^n)$ satisfies the axioms of metric.

THEOREM 3.2. *The followings are true.*

(1) *$(\mathbf{F}_p(R^n), d_p)$ and $\mathbf{F}_{c,p}(R^n))$ are complete.*
(2) *$\mathbf{F}_p(R^n)$ is the completion of $(\mathbf{F}_\infty(R^n), d_p)$.*
(3) *$\mathbf{F}_{c,p}(R^n))$ is the completion of $(\mathbf{E}^n, d_p)$.*

Proof. The proof of (1) refer to Kim [8].

(2) Let $u\in\mathbf{F}_p(R^n)$. Then $u^{(1/k)}\in\mathbf{F}_\infty(R^n)$ for each $k=1,2,\ldots$ and

$$d_p(u, u^{(1/k)}) \to 0 \text{ as } k\to\infty,$$

which implies that $\mathbf{F}_\infty(R^n)$ is a dense subspace of $(\mathbf{F}_p(R^n), d_p)$. This completes the proof. Similarly, (3) can be proved. □.

From the fact that $(\mathbf{F}_\infty(R^n), d_p)$ is separable, we can obtain the following.

COROLLARY 3.3. $(\mathbf{F}_p(R^n), d_p)$ *is separable.*

Now we can prove the following.

THEOREM 3.4. *Let* $\mathcal{U}$ *be a subset of* $(\mathbf{F}_\infty(R^n), d_p)$. *Then* $\overline{\mathcal{U}}$ *is complete if and only if* (3.3) *is true, where* $\overline{\mathcal{U}}$ *is the closure of* $\mathcal{U}$ *in* $(\mathbf{F}_\infty(R^n), d_p)$.

Proof. Suppose that (3.3) is true and $\{u_k\}$ is a Cauchy sequence in $\mathcal{U}$. Then by completeness of $(\mathbf{F}_p(R^n), d_p)$, there exists a $u \in \mathbf{F}_p(R^n)$ such that $d_p(u_k, u) \to 0$. Then there exists a decreasing sequence $\{r_i\}$ in $(0, 1]$ converging to 0 such that for each r_i

$$h(L_{r_i} u_k, L_{r_i} u) \to 0 \quad \text{as} \quad k \to \infty,$$

which implies

$$d_p(u_k^{(r_i)}, u^{(r_i)}) \leq r_i^{1/p} h(L_{r_i} u_k, L_{r_i} u) + (\int_{r_i}^1 h(L_\alpha u_k, L_\alpha u)^p \ d\alpha)^{1/p} \to 0.$$

By assumption (3.3), there exists a $v \in \mathbf{F}_\infty(R^n)$ such that $L_\alpha v = L_\alpha u^{(r_i)}$ for all $\alpha \in (r_i, 1]$ and all r_i. This implies that $u = v$, and so $u \in \mathbf{F}_\infty(R^n)$. Hence, $\mathcal{U}$ is complete.

Conversely, suppose that that $\overline{\mathcal{U}}$ is complete. Let $\{r_i\}$ be a decreasing sequence in $(0, 1]$ converging to 0 and $\{u_k\}$ be a sequence in $\mathcal{U}$ such that $\{u_k^{(r_i)}\}$ converges to $v_{r_i} \in \mathbf{F}_\infty(R^n)$ in d_p for each r_i.

Let $k_1 < k_2 < \ldots$ be a sequence of natural numbers such that

$$d_p(u_k^{(r_i)}, v_{r_i}) < \frac{1}{i} \ \text{ for } \ k \geq k_i.$$

Then by Minkowski's inequality, we have

$$\begin{aligned}
& (\int_{r_i}^1 h(L_\alpha u_{k_m}, L_\alpha u_{k_l})^p \ d\alpha)^{1/p} \\
\leq \ & (\int_{r_i}^1 h(L_\alpha u_{k_m}, L_\alpha v_{r_i})^p \ d\alpha)^{1/p} + (\int_{r_i}^1 h(L_\alpha u_{k_l}, L_\alpha v_{r_i})^p \ d\alpha)^{1/p} \\
\leq \ & d_p(u_{k_m}^{(r_i)}, v_{r_i}) + d_p(u_{k_l}^{(r_i)}, v_{r_i}) \\
< \ & \frac{2}{i} \ \text{ for } \ m, l \geq i.
\end{aligned}$$

This implies that $\{u_{k_m}\}$ is a Cauchy subsequence of $\{u_k\}$. Since $\overline{\mathcal{U}}$ is complete, $\{u_{k_m}\}$ converges to v for some $v \in \overline{\mathcal{U}}$. Then by the same arguments in the proof of Theorem 3.2 in Wu and Zhao [10], we can obtain $L_\alpha v = L_\alpha v_{r_i}$ for all $\alpha \in (r_i, 1]$ and r_i. $\square$

Now we deal with the characterization of relative compactness in $(\mathbf{F}_p(R^n), d_p)$ in a similar form as in Theorem 3.1.

THEOREM 3.5. *Let $\mathcal{U}$ be a subset of $(\mathbf{F}_p(R^n), d_p), 1 \le p < \infty$. Then $\mathcal{U}$ is relatively compact if and only if (3.1) and (3.2) are true.*

To prove this, we need the following lemmas.

LEMMA 3.6. (3.1) *implies* $\sup_{u\in\mathcal{U}} \|L_\alpha u\| < \infty$ *for each* $0 < \alpha \le 1$.

Proof. If $\sup_{u\in\mathcal{U}} \|L_\beta u\| = \infty$ for some $0 < \beta \le 1$, then for any positive natural number k, there exists $u_k \in \mathcal{U}$ such that

$$\|L_\beta u_k\| > k/\beta^{1/p}.$$

Then

$$\begin{aligned}(\int_0^1 \|L_\alpha u_k\|^p \, d\alpha)^{1/p} &\ge (\int_0^\beta \|L_\alpha u_k\|^p \, d\alpha)^{1/p} \\ &\ge (\int_0^\beta (k/\beta^{1/p})^p \, d\alpha)^{1/p} = k,\end{aligned}$$

which contradicts to (3.1). □

LEMMA 3.7. (3.1) *and* (3.2) *imply*

$$\lim_{\delta\to 0} \sup_{u\in\mathcal{U}} \int_0^\delta \|L_\alpha u\|^p \, d\alpha \ = \ 0.$$

Proof We first note that for $0 < \delta < \frac{1}{2}$,

$$\begin{aligned}&(\int_0^\delta \|L_\alpha u\|^p \, d\alpha)^{1/p} \\ \le\ &(\int_0^\delta h(L_\alpha u, L_{\alpha+\delta} u)^p \, d\alpha)^{1/p} + (\int_0^\delta \|L_{\alpha+\delta} u\|^p \, d\alpha)^{1/p} \\ =\ &(\int_\delta^{2\delta} h(L_\alpha u, L_{\alpha-\delta} u)^p \, d\alpha)^{1/p} + (\int_\delta^{2\delta} \|L_\alpha u\|^p \, d\alpha)^{1/p}.\end{aligned}$$

Similarly,

$$\begin{aligned}&(\int_\delta^{2\delta} \|L_\alpha u\|^p \, d\alpha)^{1/p} \\ \le\ &(\int_{2\delta}^{3\delta} h(L_\alpha u, L_{\alpha-\delta} u)^p \, d\alpha)^{1/p} + (\int_{2\delta}^{3\delta} \|L_\alpha u\|^p \, d\alpha)^{1/p},\end{aligned}$$

and so

$$\begin{aligned}
& (\int_0^{\delta} \|L_\alpha u\|^p \ d\alpha)^{1/p} \\
\leq \ & (\int_{\delta}^{3\delta} h(L_\alpha u, L_{\alpha-\delta}u)^p \ d\alpha)^{1/p} + (\int_{2\delta}^{3\delta} \|L_\alpha u\|^p \ d\alpha)^{1/p}.
\end{aligned}$$

By repeating this process k-times until $\frac{1}{2} \leq k\delta < (k+1)\delta \leq 1$, we can obtain

$$\begin{aligned}
& (\int_0^{\delta} \|L_\alpha u\|^p \ d\alpha)^{1/p} \\
\leq \ & (\int_{\delta}^{k\delta} h(L_\alpha u, L_{\alpha-\delta}u)^p \ d\alpha)^{1/p} + (\int_{k\delta}^{(k+1)\delta} \|L_\alpha u\|^p \ d\alpha)^{1/p} \\
\leq \ & (\int_{\delta}^{1} h(L_\alpha u, L_{\alpha-\delta}u)^p \ d\alpha)^{1/p} + \delta^{1/p}\|L_{1/2}u\|.
\end{aligned}$$

Since $\sup_{u\in\mathcal{U}} \|L_{1/2}u\| < \infty$ by Lemma 3.6, it follows that

$$\begin{aligned}
& \sup_{u\in\mathcal{U}}(\int_0^{\delta} \|L_\alpha u\|^p \ d\alpha)^{1/p} \\
\leq \ & \sup_{u\in\mathcal{U}}(\int_{\delta}^{1} h(L_\alpha u, L_{\alpha-\delta}u)^p \ d\alpha)^{1/p} + \ \delta^{1/p} \sup_{u\in\mathcal{U}} \|L_{1/2}u\| \\
\to 0 \quad & \text{as} \ \ \delta \to 0.
\end{aligned}$$

□

Proof of Theorem 3.5 : (The necessity) Let $\mathcal{U}$ be relatively compact. Then (3.1) follows immediately from the fact that $\mathcal{U}$ is bounded in $(\mathbf{F}_p(R^n), d_p)$. The proof of (3.2) is similar to the proof of Proposition 8.3.3 in Diamond and Kloeden [2].

(The sufficiency) Since $(\mathbf{F}_p(R^n), d_p)$ is complete, it suffices to prove that $\mathcal{U}$ is totally bounded.

First we note that (3.1) implies that $\{L_\alpha u : u \in \mathcal{U}\}$ is relatively compact in $(\mathbf{K}(R^n), h)$ for each $0 < \alpha \leq 1$ by Lemma 3.6 and Proposition 2.4.3 of Diamond and Kloeden [2].

Now let $\epsilon > 0$ be given. By (3.2) and Lemma 3.7, we can choose $\delta > 0$ sufficiently small so that for all $u \in \mathcal{U}$,

$$\int_0^{\delta} \|L_\alpha u\|^p \ d\alpha \quad < \quad (\epsilon/4)^p/4, \tag{3.1}$$

$$\int_{\delta}^{1} h(L_\alpha u, L_{\alpha-\delta}u)^p \ d\alpha \quad < \quad (\epsilon/2)^p/4. \tag{3.2}$$

Since $(\mathbf{K}(R^n), h)$ is complete and $\mathcal{K}_\delta = \{L_\alpha u : u \in \mathcal{U}, \delta \le \alpha \le 1\}$ is relatively compact subset of $(\mathbf{K}(R^n), h)$ by the above statement, $\mathcal{K}_\delta$ is totally bounded. That is, there exists a finite set $B_1, \cdots, B_m$ such that if $u \in \mathcal{U}, \delta \le \alpha \le 1$, then

$$h(L_\alpha u, B_i) < \epsilon \ \text{ for some } \ i.$$

Let $\delta = \alpha_0 < \alpha_1 < \cdots < \alpha_r = 1$ be a partition of $[\delta, 1]$ satisfying $\alpha_i - \alpha_{i-1} < \delta$ for all i. Now let $\mathcal{F}_\delta$ be the family of fuzzy sets v which there exists a finite unions $A_0 \supset \cdots \supset A_{r-1}$ of sets $B_1, \cdots, B_m$ such that

$$v(x) = \sum_{i=1}^{r-1} \alpha_{i-1} I_{A_{i-1} \setminus A_i}(x) + I_{A_{r-1}}(x).$$

Then $\mathcal{F}_\delta$ is finite. For $u \in \mathcal{U}$, we take $B_1, \cdots, B_r$ so that

$$h(L_{\alpha_i} u, B_i) < \epsilon / 2^{1+1/p} \ \text{ for each } \ i.$$

and let

$$A_i = \cup_{j=i}^{r} B_j.$$

Then

$$h(L_{\alpha_i} u, A_i) < \epsilon / 2^{1+1/p} \ \text{ for each } \ i.$$

If we define

$$v(x) = \sum_{i=1}^{r-1} \alpha_{i-1} I_{A_{i-1} \setminus A_i}(x) + I_{A_{r-1}}(x),$$

then

$$L_\alpha v = \begin{cases} A_0 & \text{i}f \quad\quad 0 < \alpha \le \alpha_1, \\ A_{i-1} & \text{i}f \quad \alpha_{i-1} < \alpha \le \alpha_i, \ i = 2, \cdots, r. \end{cases}$$

Thus,

$$\begin{aligned} \int_0^\delta h(L_\alpha u, L_\alpha v)^p \, d\alpha &= \int_0^\delta h(L_\alpha u, A_0)^p \, d\alpha \\ &\le \int_0^\delta [h(L_\alpha u, L_\delta u) + \epsilon/2^{1+1/p}]^p \, d\alpha \\ &\le 2^p [\int_0^\delta h(L_\alpha u, L_\delta u)^p \, d\alpha + (\epsilon/2^{1+1/p})^p \, \delta] \\ &\le 4^p \int_0^\delta \|L_\alpha u\|^p \, d\alpha + \epsilon^p \delta / 2 \\ &< \epsilon^p/4 + \epsilon^p \delta/2 \quad \text{by (1)} \end{aligned}$$

and

$$\begin{aligned}\int_\delta^1 h(L_\alpha u, L_\alpha v)^p \, d\alpha &= \sum_{i=1}^r \int_{\alpha_{i-1}}^{\alpha_i} h(L_\alpha u, L_\alpha v)^p \, d\alpha \\ &= \sum_{i=1}^r \int_{\alpha_{i-1}}^{\alpha_i} h(L_\alpha u, A_{i-1})^p \, d\alpha \\ &\le \sum_{i=1}^r \int_{\alpha_{i-1}}^{\alpha_i} [h(L_\alpha u, L_{\alpha_{i-1}} u) + \epsilon/2^{1+1/p}]^p \, d\alpha \\ &\le 2^p \sum_{i=1}^r [\int_{\alpha_{i-1}}^{\alpha_i} h(L_\alpha u, L_{\alpha_{i-1}} u)^p \, d\alpha \\ &\quad +(\epsilon/2^{1+1/p})^p(\alpha_i - \alpha_{i-1})] \\ &\le 2^p \sum_{i=1}^r [\int_{\alpha_{i-1}}^{\alpha_i} h(L_\alpha u, L_{\alpha-\delta} u)^p \, d\alpha \\ &\quad +(\epsilon/2^{1+1/p})^p(\alpha_i - \alpha_{i-1})] \\ &\le 2^p \int_\delta^1 h(L_\alpha u, L_{\alpha-\delta} u)^p \, d\alpha + \epsilon^p(1-\delta)/2 \\ &< \epsilon^p/4 + \epsilon^p(1-\delta)/2 \ \text{ by (2).}\end{aligned}$$

Therefore, we conclude that

$$d_p^p(u,v) = \int_0^\delta h(L_\alpha u, L_\alpha v)^p \, d\alpha + \int_\delta^1 h(L_\alpha u, L_\alpha v)^p \, d\alpha < \epsilon^p.$$

This completes the proof. □

Now for $u \in \mathbf{F}_p(R^n)$ and $0 < \delta < 1$, let us denote

$$\phi_p(u,\delta) = \int_0^\delta \|L_\alpha u\|^p \, d\alpha + \int_\delta^1 h(L_\alpha u, L_{\alpha-\delta} u)^p \, d\alpha.$$

Then Theorem 3.5 can be stated as follows by Lemma 3.7.

Corollary 3.8. *Let $\mathcal{U}$ be a subset of $(\mathbf{F}_p(R^n), d_p), 1 \le p < \infty$. Then $\mathcal{U}$ is relatively compact if and only if (3.1) is true and*

(3.4) $\lim_{\delta\to 0} \sup_{u\in\mathcal{U}} \phi_p(u,\delta) \ = \ 0.$

Finally, we concern with the characterization of compact sets on $\mathbf{F}_\infty(R^n)$ relative to the sendograph metric D. The followings were obtained by Greco [4], Zhao and Wu [12].

Theorem 3.9. ([4]) *Let $\mathcal{U} \subset \mathbf{F}_\infty(R^n)$. Then $\mathcal{U}$ is relatively compact relative to D if and only if*

(3.5) *$\mathcal{U}$ is uniformly support bounded, i.e.,* $\sup_{u\in\mathcal{U}} \|u\| < \infty$;
(3.6) *$\mathcal{U}$ is equi-right-continuous at* 0*, i.e.,*

$$\lim_{\alpha\to 0} \sup_{u\in\mathcal{U}} h(L_\alpha u, L_0 u) = 0.$$

THEOREM 3.10. ([12]) *Let $\mathcal{U} \subset \mathbf{F}_\infty(R^n)$. Then $\mathcal{U}$ is relatively compact relative to D if and only if $\mathcal{U}$ is equi-right-continuous at* 0 *and*

(3.7) *$\mathcal{U}$ is relatively compact relative to d_p for any $1 \le p < \infty$.*

The above theorem can be improved as follows;

THEOREM 3.11. *Let $\mathcal{U} \subset \mathbf{F}_\infty(R^n)$. Then $\mathcal{U}$ is relatively compact relative to D if and only if $\mathcal{U}$ is equi-right-continuous at* 0 *and*

(3.8) *$\mathcal{U}$ is uniformly p-th mean bounded for some $1 \le p < \infty$.*

Proof. The necessity follows immediately from Theorem 3.9 because (3.5) implies that $\mathcal{U}$ is uniformly p-th mean bounded.

To prove the sufficiency, we note that $\sup_{u\in\mathcal{U}} \|L_\alpha u\| < \infty$ for any $0 < \alpha \le 1$ by Lemma 3.7. Since $\mathcal{U}$ is equi-right-continuous at 0, we can choose $0 < \delta \le 1$ so that

$$h(L_\delta u, L_0 u) < 1 \text{ for all } u \in \mathcal{U}.$$

And then, the inequality

$$\|L_0 u\| \le h(L_0 u, L_\delta u) + \|L_\delta u\|$$

implies that $\mathcal{U}$ is uniformly support bounded. This completes the proof.

4. Conclusions

In this paper, we established a characterization of relatively compact subsets of fuzzy sets in R^n relative to L_p-metric. Also, we improved relative compactness criteria on the space of fuzzy numbers relative to the sendograph metric. These results will play an important role in the further research of fuzzy analysis and fuzzy random variables.

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